The Nash problem and its solution

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Abstract

The goal of this paper is to give a historical overview of the Nash Problem of arcs in arbitrary dimension, as well as its affirmative solution in dimension two by J. Fernandez de Bobadilla and M. Pe Pereira and a negative solution in higher dimensions by T. de Fernex, S. Ishii and J. Kollár. This problem was stated by J. Nash around 1963 and has been an important subject of research in singularity theory.

1 Introduction

In this paper, $\Bbbk$ is an algebraically closed field of characteristic 0 (see Remark 1.8 below for the case of positive characteristic).

1.1 Resolution of singularities

Let X be a singular algebraic variety over $\Bbbk$ and $\pi : \tilde{X} \longrightarrow X$ a *divisorial* resolution of singularities of X (this means that π is bijective away from the closed set $\pi^{-1}(Sing\ X)$ — such morphisms are called **birational** — $\tilde{X}$ is a smooth variety and the **exceptional set** $E =: \pi^{-1}(Sing\ X)$ is a **divisor**, that is, is of pure codimension one). Let

$$E = \bigcup_{i \in \Delta} E_i \tag{1}$$

be the decomposition of E into its irreducible components. The set E has two kinds of irreducible components: essential and inessential. Intuitively, an irreducible divisor is essential if it appears, as an irreducible divisor, on *every* divisorial resolution of X.

In general (that is, when dim $X \geqslant 3$) it is quite difficult to show that a given component is essential (see [31] for a discussion of this question as well as some sufficient conditions for essentiality and [3] and [16] for new criteria of essentiality). In dimension two there exists a unique *minimal* resolution $\tilde{X}$ of X (in the sense that any other resolution of X maps to $\tilde{X}$) and each irreducible exceptional divisor of $\tilde{X}$ is essential.

Example 1.1

- *The first example is the following (see Ishii-Kollar [15]): Let X be defined by $xy - uv = 0$ in $\mathbb{C}^4$. The variety X is of dimension 3, with isolated singularity at 0. One can resolve it either by blowing up up the point 0 (the map P_2 on the figure) or by blowing up the surface on X defined by $x = u = 0$ (the map P_1). In the first case, one obtains a divisorial resolution with one divisor, in the second case, the exceptional set E is a curve, hence of codimension 2, and so is not a divisor. Thus in dimension higher than 2, not all the resolutions are divisorial. The second resolution is an example of a* **small resolution**, *that is a resolution in which every irreducible component of the exceptional set has codimension strictly greater than 1.*

- *The second example is the variety defined in $\mathbb{C}^4$ by*

$$x^2 + y^2 + z^2 + w^4 = 0.$$

This variety can be resolved by two blowing ups at 0, this resolution being divisorial. But there also exists a small resolution, given by only one blowing up the subvariety defined by $x - y = 0 = z - w^2$, which gives only one component for E. Thus one of the two divisors found in the first resolution is not essential.

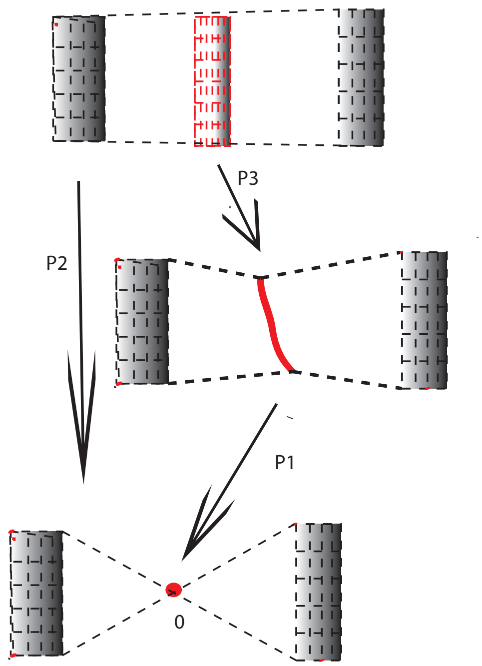

Figure 1 : Two resolutions of xy-uv=0

1.2 The space of arcs of X

In order to study resolutions $\tilde{X}$ of X, J. Nash (around 1963, published in 1995 [25]) introduced the space X_∞^{sing} of *arcs* meeting the singular locus *Sing* X.

To give an idea of what the space X_∞^{sing} and its elements look like, let us first take $\Bbbk = \mathbb{C}$ and consider the space of all the germs of parametrized *analytic* curves, contained in the algebraic variety X over $\mathbb{C}$ and meeting the singular locus *Sing* X. For example, suppose X is an affine variety, defined in $\mathbb{C}^N$ by polynomial equations $f_1, \dots, f_s$ in N variables. By definition, a parametrized analytic arc in X, meeting *Sing* X, is given by N convergent power series

$$\left\{ \begin{array}{c} x_1(t) = a_{10} + a_{11}t + a_{12}t^2 + \dots \\ x_2(t) = a_{20} + a_{21}t + a_{22}t^2 + \dots \\ \vdots \\ x_N(t) = a_{N0} + a_{N1}t + a_{N2}t^2 + \dots \end{array} \right. \tag{2}$$

having the following properties:

(a) for all $j \in \{1, \dots, s\}$, the convergent power series $f_j(x_1(t), \dots, x_N(t))$ is identically zero as a power series in t

(b) the point $(a_{10}, \dots, a_{N0})$, which we refer to as **the origin** of the arc (2), belongs to *Sing* X.

A specific example of this situation when X is the hypersurface in $\mathbb{C}^3$ defined by the equation $xy - z^{n+1} = 0$ is discussed in Example 1.3 below.

For the purposes of the Nash problem, it is natural to consider *formal* parametrized curves instead of analytic ones, that is, to drop the convergence assumption on the power series $x_1(t), \dots, x_N(t)$ above. In the algebraic language, we say that the power series (2) define a morphism from *Spec* $\mathbb{C}[[t]]$ to X such that the image of the closed point of *Spec* $\mathbb{C}[[t]]$ belongs to *Sing* X. Once the definition is expressed in the algebraic language, it is natural to extend it to arbitrary algebraically closed fields $\Bbbk$ and to varieties X which are not necessarily affine:

Definition 1.2 *An **arc** is a $\mathbb{k}$-morphism from Spec $\mathbb{k}[[t]]$ to X.*
Let X_∞^{sing} be the set of arcs whose origin (that is, the image of the closed point) belongs to the singular locus of X.

The analogue of an arc in complex analysis is a test map from a small disc around the origin on the complex plane to X. We will also need to consider more general arcs, which are morphisms from *Spec* $K[[t]]$ to X, where K is a field extension of $\mathbb{k}$; they are called K-**arcs**.

Example 1.3 *Let us have a look at the singularity A_n given in $\mathbb{C}^3$ by the equation*

$$z^{n+1} = x.y$$

It is the first example studied by J.Nash. It has an isolated singularity at 0.

An arc living on A_n and passing through 0 *is given by three formal power series*

$$\begin{cases} x(t) = a_1 t + a_2 t^2 + \ldots \\ y(t) = b_1 t + b_2 t^2 + \ldots \\ z(t) = c_1 t + c_2 t^2 + \ldots \end{cases}$$

whose coefficients are elements of $\mathbb{C}$ and such that $z(t)^{n+1} \equiv x(t).y(t)$. That last equation gives an infinity of equations on the coefficients of the arcs:

$$\begin{cases} a_1 b_1 = 0 \\ a_1 b_2 + a_2 b_1 = 0 \\ \vdots \\ a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_{n-1} b_1 \\ c_1^{n+1} = a_1.b_n + \ldots + b_1 a_n \\ \vdots \end{cases}$$

Let us denote the closed point (the origin) of *Spec* $\mathbb{k}[[t]]$ by 0 and the generic point by η.

An arc can be lifted to any resolution:

Lemma 1.4 *Let $f : \tilde{X} \to X$ be a resolution of singularities. Every arc α : Spec $K[[t]] \to X$ such that $\alpha_x(\eta) \notin Sing(X)$ can be lifted uniquely to an arc $\tilde{\alpha}$: Spec $\mathbb{k}[[t]] \to \tilde{X}$.*

The proof comes from the fact that the resolution map π is proper. In other words, as the resolution of singularities is an isomorphism away from E, one can lift the arc without the origin, and then take the closure.

Remark: the closure of each lifted arc intersects at least one of the irreducible exceptional divisors; moreover if an arc is general enough, its lifting intersects transversely one and only one irreducible exceptional divisor.

Let us fix a divisorial resolution of singularities $\tilde{X} \to X$ and let $E = \pi^{-1}(Sing\ X)$. Consider the decomposition (1) of E into irreducible components, as above. Let $\Delta' \subset \Delta$ denote the set which indexes the essential divisors.

M. Lejeune-Jalabert [19], inspired by Nash's original paper [25], proposed the following decomposition of the space X_∞^{sing}: for $i \in \Delta'$, let C_i be the set of arcs whose lifting in $\tilde{X}$ intersects the essential divisor E_i transversally but does not intersect any other exceptional divisor E_j. M. Lejeune-Jalabert shows that

$$X_\infty^{sing} = \bigcup_{i \in \Delta'} \overline{C_i} \tag{3}$$

and the set $\overline{C_i}$ is an irreducible algebraic subvariety of the space of arcs.

1.3 The statement of the problem

Nash used the decomposition (3) to show that X_∞^{sing} has finitely many irreducible components, $F_1, \ldots, F_r$, called *families of arcs*, and defined the following map:

Definition 1.5 (Nash [25]) *Let*

$$\mathcal{N} : \{F_1, \ldots, F_r\} \to \{ \text{ essential divisors } \}$$

be the map sending the family F_i to the exceptional divisor E_i such that the generic arc of F_i has lifting to the resolution passing through a general point of the component E_i.

He showed that this map, now called the **Nash map**, is injective. The celebrated **Nash problem**, posed in [25], is the question whether the Nash map is surjective.

Since the $\overline{C_i}$ are irreducible, the families of arcs are among the $\overline{C_i}$'s. Moreover there are as many $\overline{C_i}$ as essential divisors E_i. Then the Nash problem reduces to showing that the $\overline{C_i}$, $i \in \Delta'$, are precisely the irreducible components of X_∞^{sing}, that is, to proving $card(\Delta')(card(\Delta') - 1)$ non-inclusions:

Problem 1.6 *Is it true that $\overline{C_i} \not\subset \overline{C_j}$ for all $i \neq j$?*

Example 1.7 *Let us keep our attention on the singularity A_n given in $\mathbb{C}^3$ by the equation*

$$z^{n+1} = x.y$$

It has an isolated singularity at 0. The exceptional divisor of the minimal resolution of A_n consists of n irreducible curves E_i, arranged in a chain, such that E_i intersects E_{i+1} transversely for $i \in \{1, \ldots, n-1\}$.

One can show that the first n equations of Example 1.3 completely describe the n irreducible components of $A_{n,\infty}^{sing}$ and that the irreducible component F_i is given by the $a_1 = ... = a_{i-1} = b_1 = ... = b_{n-i} = 0$. A general element of F_k has the form :

$$\begin{cases} x(t) = a_k t^k + a_{k+1} t^{k+1} + \ldots \\ y(t) = b_{n+1-k} t^{n+1-k} + b_{n+2-k} t^{n+2-k} + \ldots \\ z(t) = c_1 t + c_2 t^2 + \ldots \end{cases}$$

with a_k, b_{n+1-k} and c_1 different from zero.

Remark 1.8 *All of the above definitions make sense also when car $\mathbb{k} > 0$, with the following modification. An arc family is said to be* **good** *if its general element is not entirely contained in Sing X. When char $\mathbb{k} = 0$ it is easy to show that all the arc families are good. Over fields of positive characteristic there may exist some bad families, and the Nash map is only defined on the set of good families. With this in mind, the Nash problem remains the same: is the Nash map, defined on the set of good families, surjective? See [36] for some recent work on the Nash problem in positive characteristic.*

1.4 Some partial answers in dimension 2

Before the work of Fernandez de Bobadilla — Pe Pereira, the Nash problem for surfaces has been answered affirmatively in the following special cases: for A_n singularities by Nash, for minimal surface singularities by A. Reguera [33] (with other proofs by J. Fernandez-Sanchez [6] and C. Plénat [28]), for sandwiched singularities by M. Lejeune-Jalabert and A. Reguera (cf. [20] and [34]), for toric vareties in all dimensions by S. Ishii and J. Kollar [15] (using earlier work of C. Bouvier and G. Gonzalez-Sprinberg [1] and [2]), for a family of non-rational surface singularities by P. Popescu-Pampu and C. Plénat ([30]), for quotients of $\mathbb{C}^2$ by an action of finite group [26] by M. Pe Pereira in 2010 based on the work [4] of J. Fernandez de Bobadilla (other proofs for $\mathbf{D_n}$ in 2004 by Plénat [29], for E_6 in 2010 by C. Plénat and M. Spivakovsky [32], with a method that works for some normal hypersurface singularities), and by M. Leyton-Alvarez

(2011) for E_6 and E_7, by applying the method for the following classes of normal hypersurfaces in $\mathbb{C}^3$: hypersurfaces $S(p, h_q)$ given by the equation $z^p + h_q(x, y) = 0$, where h_q is a homogeneous polynomial of degree q without multiple factors, and $p \geqslant 2$, $q \geqslant 2$ are two relatively prime integers [22]. A. Reguera [36] gave an affirmative answer to the Nash problem for rational surface singularities simultaneously and independently from the work [5].
See the bibliography for a (hopefully) complete list of references on the subject.

In 2011, J. Fernandez de Bobadilla and M. Pe Pereira [5] showed that the answer is positive for any surface singularity. The main aim of this paper is to give an idea of their proof. Before going further into the details, we need to recall some earlier results that lead to the final proof.

The rest of the paper is organized as follow: §2 is dedicated to the work preceding the paper [5]; in §3 an outline of the proof is given. §4 contains a brief discussion of the Nash problem in dimension three and higher.

2 The wedge problem.

2.1 The Wedge problem [17] ...

In 1980, M.Lejeune-Jalabert proposed to look at the Nash problem from a new point of view. She formulated in [17] what is now called "the wedge problem", which is related to a "Curve Selection Lemma" in the space of arcs.

Let X be a singular algebraic variety over $\mathbb{k}$.
Let us first define **wedge**:

Definition 2.1 *Let K be a field extension of $\mathbb{k}$. A K-wedge on X is a $\mathbb{k}$-morphism*

$$\omega : Spec(K[[t, s]]) \to X$$

which maps the set $\{t = 0\}$ to $Sing\ X$.

The wedge ω induces two arcs on X as follows: a K-arc obtained by restricting ω to the set $\{s = 0\}$ (this arc is called the **special arc** of ω), and a $K((s))$-arc, obtained by restricting ω to the set $Spec(K[[t, s]]) \setminus \{s = 0\}$ (this arc is called the **general arc** of ω). We regard ω as a deformation of its special arc to its general arc or, alternatively, as an arc in the space of arcs X_∞^{sing}.

The wedge is said to be **centered** at an arc γ_0 if its special arc is γ_0.
Let $(X, 0)$ be a germ of a normal surface singularity, and let $\pi : (\tilde{X}, E) \to (X, 0)$ be its minimal (and so divisorial) resolution, with $E = \bigcup E_j = \pi^{-1}(0)$. Let E_i, E_j be irreducible components of E (they are essential as X is a surface). Let C_i and C_j be as above. Then if $C_j \subset \overline{C_i}$, E_j is not in the image of Nash map. If one had Curve Selection lemma in the space of arcs X_∞^{sing}, the inclusion above would just mean that one has a $\mathbb{k}$-wedge with special arc in C_j and generic arc in C_i. Then the morphism ω would not lift to the resolution $\tilde{X}$ as it has an indeterminacy at 0.

M. Lejeune-Jalabert proposed the following problem:

Problem 2.2 *For all irreducible essential divisors of the minimal resolution, any $\mathbb{k}$-wedge centered at $\gamma_i \in C_i$ can be lifted to $\tilde{X}$.*

It is not trivial to generalize the classical Curve Selection Lemma to the case of infinite-dimensional varieties such as X_∞^{sing}. A. Reguera proved a Curve Selection Lemma for X_∞^{sing} thus establishing the equivalence between the the Nash and the wedge problems. The wedges appearing in A. Reguera's theorem are K-wedges rather than $\mathbb{k}$-wedges, where K is an extension of $\mathbb{k}$ of infinite transcendence degree. In the following section we discuss this work of A. Reguera and its generalizations due to J. Fernandez de Bobadilla and A. Reguera – M. Lejeune-Jalabert, which reduce the Nash problem to the problem of lifting of $\Bbbk$-wedges to the minimal resolution.

2.2 ...is equivalent to Nash problem of arcs

In the paper [35], A. Reguera has shown that a positive answer to the wedge problem is equivalent to the surjectivity of the Nash map. She has also extended the wedge problem to all dimensions. Note that she does not assume the singular varieties to be normal. More precisely, she proves the following:

Theorem 2.3 *Let X be a singular variety.*
Let E_i be an essential divisor over X. Let γ_i be the generic point of $\overline{C_i}$ (the closure of the set of arcs lifting transversally to E_i), $\Bbbk_i$ its residue field. The following are equivalent:

1. *E_i belongs to the image of the Nash map.*
2. *For any resolution of singularities $p : \tilde{X} \to X$ and for any field extension K of $\Bbbk_i$, any K-wedge whose special arc maps to γ_i, and whose generic arc maps to X_∞^{sing}, lifts to $\tilde{X}$.*
3. *There exists a resolution of singularities $p : \tilde{X} \to X$ satisfying the conclusion of* (2)*.*

To prove this she needed a Curve Selection lemma for X_∞^{sing} for curves defined over K. This field is of infinite transcendence degree over $\Bbbk$, so it is quite difficult to work with. J. Fernandez de Bobadilla [4] and M. Lejeune-Jalabert with A. Reguera [21] have shown, independently, that one may replace K by $\Bbbk$ in A. Reguera's theorem, provided that $\Bbbk$ is uncountable.

Let us cite some results from Fernandez de Bobadilla's paper. First, he gives the definition of wedges that realize an adjacency between two essential divisors:

Definition 2.4 *Let E_u and E_v be two essential divisors, and C_u and C_v the irreducible subvarieties of X_∞^{sing} associated to these divisors.*
A K-wedge realizes an adjacency from E_u to E_v if its generic arc belongs to C_u and its special arc belongs to $\dot{C}_v$ (i.e. it is transverse to E_v in a general point of E_v).

Note that if such a wedge exists, then C_v is not in the image of Nash map. This statement can be interpreted as the easy part of the previous Theorem of A.Reguera ($1 \implies 2$): a wedge realizing the adjacency cannot be lifted to any resolution.

J. Fernandez de Bobadilla proves the following theorem:

Theorem 2.5 *Let $(X, 0)$ be a normal surface singularity defined over an uncountable algebraically closed field $\Bbbk$ of characteristic 0. Let E_v be an essential irreducible component of the exceptional divisor of a resolution. Then the following are equivalent:*

1. *The set C_v is in the Zariski closure of C_u, where E_u is another irreducible component of the exceptional divisor.*
2. *Given any proper closed subset $\mathcal{Z} \subset \overline{C_u}$, there exists an algebraic $\Bbbk$-wedge realizing an adjacency from E_u to E_v and avoiding $\mathcal{Z}$.*
3. *There exists a formal $\Bbbk$-wedge realizing an adjacency from E_u to E_v.*
4. *Given any proper closed subset $\mathcal{Z} \subset \overline{C_u}$, there exists a finite morphism realizing an adjacency from E_u to E_v and avoiding $\mathcal{Z}$.*

See [4] for the definition of finite morphism realizing an adjacency from E_u to γ.

J. Fernandez de Bobadilla also proved in [4] that the Nash problem for surfaces is a topological problem, in other words, if the answer is affirmative for a certain normal surface singularity $(X, 0)$, it is also positive for any normal surface singularity, diffeomorphic to X:

Theorem 2.6 *The set of adjacencies between exceptional divisors of a normal surface singularity is a combinatorial property of the singularity: it only depends on the dual weighted graph of the minimal good resolution. In the complex analytic case this means that the set of adjacencies only depends on the topological type of the singularity and not on the complex structure.*

A sketch of the proof for normal dimension two singularities is the subject of the following section. We will need to first define what a geometric Milnor representative of an arc and a wedge are.

3 Solution of the Nash problem for surfaces

Theorem 3.1 *Let $\mathbb{k}$ be an algebraically closed field of characteristic 0 and $(X, 0)$ a normal singular surface over $\mathbb{k}$.*
The Nash map associated to $(X, 0)$ is bijective.

In [4] (7.2 p. 163), J. Fernandez de Bobadilla shows that the families of arcs are stable under base change and so is the bijectivity of Nash map. This allows to reduce the problem to the case of normal surface singularities over $\mathbb{C}$.

Let $(X, 0)$ be a normal surface singularity over $\mathbb{C}$. (The non-normal case can be reduced to the normal one).

The proof proceeds by contradiction.

Let $E = \bigcup_{i=0}^{n} E_i$ be the decomposition of E into irreducible components. Suppose there are two irreducible subvarieties of X_∞^{sing} $\overline{C_0}$ and $\overline{C_i}$ associated with two essential divisors E_0 and E_i of the minimal resolution such that $\overline{C_0} \subset \overline{C_i}$.

From now on, replace X by its underlying complex-analytic space. By abuse of notation, we will continue to denote this space by X. Let $\pi : \tilde{X} \to X$ be the minimal resolution of X.

For an analytic wedge $\alpha : (\mathbb{C}^2, 0) \to X$ we denote the generic arc by $\alpha(t, s) = \alpha_s(t)$ and the special arc $\alpha(t, 0)$ by $\gamma(t)$. Aiming for contradiction, we now consider an analytic wedge $\alpha : (\mathbb{C}^2, 0) \to X$ realizing the adjacency from E_i to E_0, that is, a wedge such that the generic arc belongs to C_i and the special arc belongs to C_0.

J. Fernandez de Bobadilla and M. Pe Pereira define the notion of Milnor representative of arcs and wedges.

Let us call $X_{\varepsilon_0} = X \cap B_{\varepsilon_0}$ the Milnor representative of X. This means, by definition, that for all $0 < \varepsilon \leqslant \varepsilon_0$ the sphere S_ε is transverse to X and $X \cap S_\varepsilon$ is a closed subset of S_ε.

Consider the special arc $\gamma : (\mathbb{C}, 0) \to X_{\varepsilon_0}$. It is proved in [26] and [5] that there exists $\varepsilon \leqslant \varepsilon_0$ such that, restricted to X_ε, γ becomes a **Milnor arc**:

Definition 3.2 Milnor arc
A Milnor representative of γ is a map of the form

$$\gamma|_U : U \to X_\varepsilon$$

such that $\gamma|_U$ is a proper morphism, U is diffeomorphic to a closed disc, $\gamma^{-1}(\partial X_\varepsilon) = \partial U$ and the mapping $\gamma_{|U}$ is transverse to any sphere $S_{\varepsilon'}$ for $\varepsilon' \leqslant \varepsilon$. The radius ε is called a Milnor radius for γ.

Let $\gamma\,|_U : U \to X_\varepsilon$ be a Milnor Representative of γ.

For the disc D_δ of radius δ around the origin in the complex plane we will use the notation $\dot{D}_\delta = D_\delta \setminus \{0\}$.

We replace α by its restriction to $U \times D_\delta$, where δ is a small positive number, specified immediately below.

Milnor wedge.
There exist $\delta > 0$ small enough, an open set $\mathcal{U} \subset U \times D_\delta$ and a map

$$\beta : U \times D_\delta \to X_\varepsilon \times D_\delta$$
$$(t,s) \quad \to (\alpha_s(t), s)$$

such that the set $U_s = \mathcal{U} \cap \mathbb{C} \times \{s\}$ is diffeomorphic to a disc for all s and satisfying some other transversality and finiteness conditions, which we omit in order not to overburden the exposition with technical details.

Definition 3.3 *The map β restricted to $\mathcal{U}$ is a* **Milnor representative** *of the wedge α.*

Remark 3.4 *One has to prove that such a representative does exist, in particular that the set $\mathcal{U}$ can be taken to be differomorphic to a bidisk. See [26] or [5].*

These definitions of representatives are a key point in the proof of the theorem.

The main idea of the proof: Let $\alpha_s : U_s \to X_\varepsilon$ be a generic arc of the wedge. By construction, U_s is a disk and thus has Euler characteristic equal to one. The aim of the rest of the proof is to show that the Euler characteristic of U_s is bounded above by an expression less or equal to 0, and thus get a contradiction.

3.1 Eliminating the indeterminacy of $\tilde{\alpha}$

Let $\tilde{\beta}$ be the meromorphic map defined as the composition of $\sigma^{-1} \circ \beta$ with $\sigma = (\pi, id\,|_{D_\delta})$:

$$\begin{tikzcd} & \tilde{X}_\varepsilon \times D_\delta \arrow[d, "\sigma"] \\ \mathcal{U} \arrow[ur, dashed, "\tilde{\beta}"] \arrow[r, "\beta"] & X_\varepsilon \times D_\delta \end{tikzcd}$$

The indeterminacy locus of $\sigma^{-1} \circ \beta$ is of codimension 2. Thus we may assume that, shrinking the radius δ, if necessary, $(0,0)$ is the only indeterminacy point of $\tilde{\beta}$.

Moreover there exists a unique meromorphic lifting $\tilde{\alpha}$ of α such that:

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & \tilde{X}_\varepsilon \arrow[d, "\pi"] \\ \mathcal{U} \arrow[ur, dashed, "\tilde{\alpha}"] \arrow[r, "\alpha"] & X_\varepsilon \end{tikzcd}$$

Let Y be the analytic Zariski closure of $\sigma^{-1}(\beta(\mathcal{U}) \backslash (\{0\} \times D_\delta))$ and let $Y_s = Y \cap (\tilde{X}_\varepsilon \times \{s\})$. The surface Y is reduced and is a Cartier divisor in the smooth threefold $\tilde{X} \times D_\delta$. One can prove the following ([5]):

$$Y_s = \tilde{\alpha}_s(U_s) \quad \forall s \in \dot{D}_\delta.$$

Remark 3.5 *Y_s can be thought as the topological image of the lifting of the arc α_s on $\tilde{X}_\varepsilon$.*

Moreover, one has that:

Lemma 3.6 *The mapping $\tilde{\alpha}_s : U_s \to Y_s$ is the morphism of normalization of Y_s.*

To prove this, we use the following lemma :

Lemma 3.7 *The mapping $\alpha_s : U_s \to X_\varepsilon$ is one–to–one.*

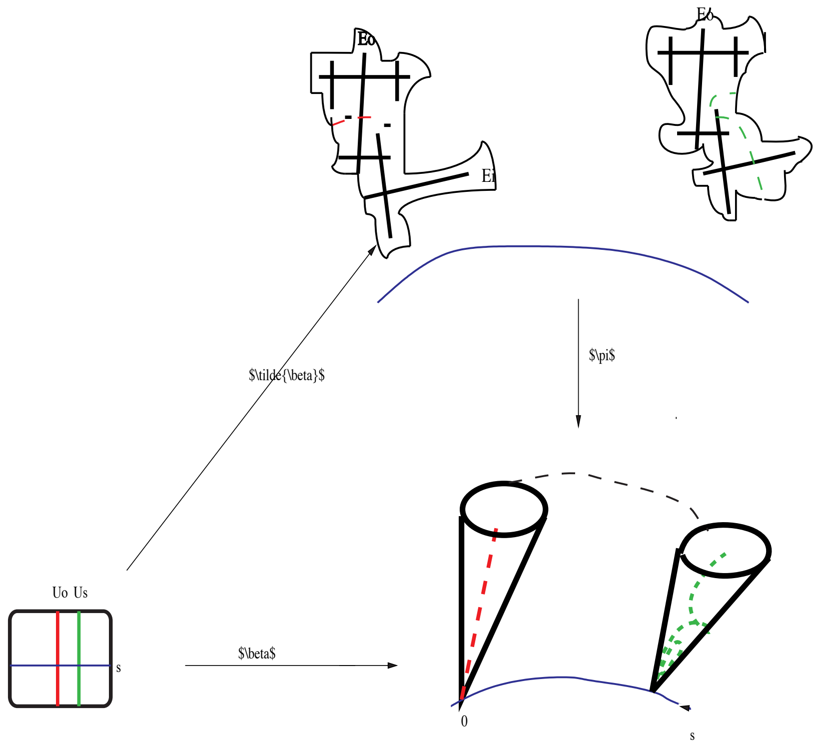

Figure 2 : Wedge representative

Definition 3.8 Returns
Elements of the set $\alpha_s^{-1}(0) \setminus \{0\}$ *are called* **returns**. *Their images by* α_s *are* 0 *and by* $\tilde{\alpha_s}$ *points of the exceptional set* E.

As explained before, to obtain a contradiction we want to show that U_s has non-positive Euler characteristic. To do this, Fernandez de Bobadilla and Pe Pereira give an upper bound on $\chi(U_s)$ in terms of $\chi(Y_s)$, $\chi(Y_0)$ and the possible returns.

3.1.1 End of the proof

The curve $Y_0 = Y \cap (\tilde{X}_\varepsilon \times \{0\})$ does not need to be reduced. It contains $Z_0 := \tilde{\alpha_0}(U_0)$ and a sum of the exceptional components E_i with suitable multiplicities. We express this situation by the equation $Y_0 = Z_0 + \sum a_i E_i$; the analytic space Y_0 is reduced along $Z_0 \backslash E$.
A crucial point in the proof of Fernandez de Bobadilla – Pe Pereira is the fact that Y_s is a deformation of Y_0, and hence is numerically equivalent to it, that is, Y_s and Y_0 have the same intersection number with any compact curve in $\tilde{X}_\varepsilon$. We construct a tubular neighborhood of E in the following way.

Define $\dot{E}_i = E_i \backslash Sing(Y_0^{red})$. Let $Sing(Y_0^{red}) = \{p_0, p_1, ..., p_m\}$, where $p_0 = Z_0 \cap E$. Let B_k be a small ball in $\tilde{X}$ centered at p_k. For $j \in \{0, \dots, n\}$, let T_j be a tubular neighborhood of E_j, small enough so that its intersection with each B_k is transverse. Let T_{n+1} be a tubular neighborhood of Z_0, small enough so that its intersection with B_0 is transverse. Let $W_j = T_j \backslash \left(\bigcup_{k=0}^{m} B_k \right)$. All the neighborhoods are chosen so that

$$\chi(U_s) = \sum_{j=0}^{n+1} \chi(\tilde{\alpha_s}^{-1}(Y_s \cap W_j)) + \sum_{k=0}^{m} \chi(\tilde{\alpha_s}^{-1}(Y_s \cap B_k)). \quad (4)$$

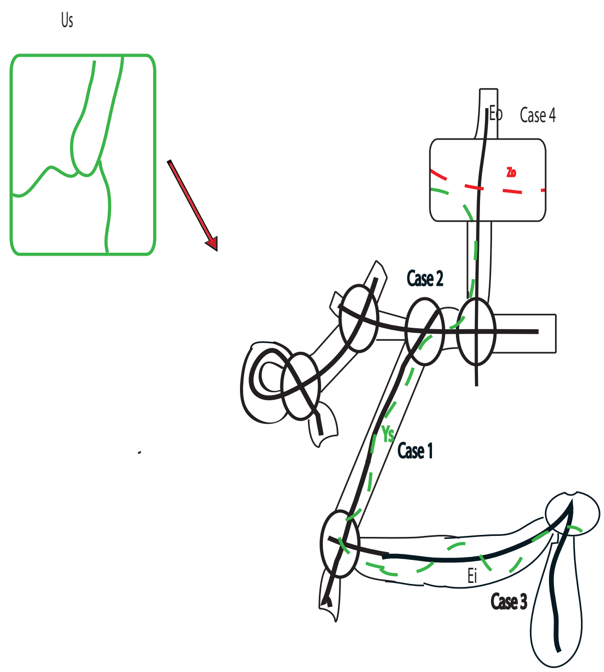

Figure 3 : Normalization map

We do not need to count $\chi(Y_s \cap T_j \cap B_k)$ since by the assumed transversality each of these intersections is a finite union of circles and thus

$$\chi(Y_s \cap T_j \cap B_k) = 0. \tag{5}$$

It remains to bound above each summand on the right hand side of (4). Using topological techniques, the authors prove, under some extra assumptions, that

$$\chi(U_s) \leqslant \sum_i a_i(2 - 2g_i + E_i.E_i). \tag{6}$$

This last sum is less or equal to 0 as each member is less than or equal to 0. This proves that the disc U_s has non-positive Euler characteristic, which gives the desired contradiction. In the general case, the authors obtain a more complicated version of the formula (6), which leads to the same final conclusion.

4 Higher dimensions

For singularities of higher dimensions, the Nash Problem stated as above is false, though a few positive results have been proved: in [15], S. Ishii and J. Kollar give an affirmative answer for toric varieties in all dimensions. Affirmative answers were given for a family of singularities in dimension higher than 2 by P. Popescu-Pampu and C. Plénat ([31]) and for another family by M. Leyton-Alvarez [22] (2011).

In [15], S. Ishii and J. Kollár give a counterexample to the Nash problem in dimension greater than or equal to 4: the hypersurface

$$x^3 + y^3 + z^3 + u^3 + w^6 = 0$$

which has a resolution with two irreducible exceptional components. These are essential, as one is the projectivization of the tangent cone at the singular point (hence it clearly corresponds to a Nash family), and the other one is not uniruled. Then the authors construct geometrically a wedge whose generic arc is in the Nash family, and whose special arc is in the second family.

In May 2012, T. de Fernex gave a counterexample in dimension 3 ([3], 2012). The equation is

$$(x^2 + y^2 + z^2)w + x^3 + y^3 + z^3 + w^5 + w^6 = 0 \tag{7}$$

In the algebraic setting, he can prove that the two exceptional components obtained after two blowing-ups are essential. But as an analytic variety, the hypersurface obtained from (7) by blowing up the origin is locally isomorphic to the non-degenerate quadratic cone, hence it admits a small resolution; this implies that the second exceptional component is not essential, so the counterexample does not apply in the analytic category. Deforming the equation (7), de Fernex obtains a counterexample to the Nash problem in dimension 3, valid in both the algebraic and the analytic setting:

$$(x^2 + y^2)w + x^3 + y^3 + z^3 + w^5 + w^6 = 0.$$

An even more recent paper on the Nash problem is due J. Kollár [16]. In that paper, J. Kollár gives a new family of counterexamples to the Nash problem in dimension 3, called $c\mathcal{A}_1$-type singularities:

$$x^2 + y^2 + z^2 + t^m = 0$$

with m odd, $m > 3$. These singularities are isolated and have only one Nash family, but two of the exceptional components in the resolution are essential.
Moreover, Kollár formulates the Revised Nash problem, which we now explain.

Definition 4.1 *Let X be a variety over a field k, $k \subset K$ a field extension of k and $\phi : Spec\ K[[t]] \to X$ an arc such that $Supp\ \phi^{-1}(Sing(X)) = \{0\}$. A* **sideways deformation** *of ϕ is an extension of ϕ to a morphism $\Phi : Spec\ K[[t,s]] \to X$ such that $Supp\ \Phi^{-1}(Sing(X)) = \{(0,0)\}$.*

Definition 4.2 *We say that X is* **arcwise Nash-trivial** *if every general arc in X_∞^{sing} has a sideways deformation.*

Definition 4.3 *Let X be a variety over k. A divisor over X is called* **very essential** *if the following holds. Let $p : Y \to X$ be a proper birational morphism such that Y is $\mathbb{Q}$-factorial and has only arcwise Nash-trivial singularities. Then $center_Y E$ is an irreducible component of $p^{-1}(Sing(X))$.*

In fact in the three counterexamples above, the components corresponding to Nash families are given precisely by the unique very essential divisor. Imitating and conceptualizing the proofs of non-essentiality appearing in the above counterexamples, one can show in full generality that divisors appearing in the image of the Nash map are always very essential. We are lead to the following problem:

Problem 4.4 *Is the Nash map surjective onto the set of very essential divisors?*

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